Uncertainty and rounding

Contents

1 Measured quantitities

Definition. *A* ***measured quantity*** *consists of:*

- *the* ***type*** *of quantity measured: length, volume, duration, temperature, number of people, price, ...*
- *the* ***unit*** *of measurement used: m, cm, inch, light-years, microseconds (μs), millenia, CHF, °F, people, ...*
- *the* ***magnitude*** *that has been measured (an integer or terminating decimal number)*
- *the (absolute)* ***uncertainty*** *associated with the measurement: this is an upper estimate of the measurement's level of precision*

Example 1.1. *The width w of a closet has been measured with a measuring band. We read on the band the magnitude 61.4, but estimate that the measurement may be off by as much as 2 mm. So the true width lies somewhere between 61.2 and 61.6 cm. We write:*

- $w = 61.4 \pm 0.2$ *cm*
- $61.2 \leq w \leq 61.6$
- $w \in [61.2, 61.6]$ *(**measurement interval**)*

Example 1.2. *In particle physics, the "fine-structure constant" α is known with impressive precision:*

$$\alpha = 0.007\,297\,352\,566\,4 \pm 0{,}000\,000\,000\,001\,7 \quad \textit{(unitless)}$$

Because this is somewhat hard to read, the uncertainty in the last two digits can also be written in parentheses behind the magnitude:

$$\alpha = 0.007\,297\,352\,566\,4(17).$$

The uncertainty can be expressed either in absolute terms (in the same unit as the magnitude) or in relative terms (as a fraction or percentage of the magnitude).

Definition. *The* ***relative uncertainty*** *of a measured quantity is the ratio between its absolute uncertainty ΔQ and its magnitude Q:*

$$\delta Q = \frac{\Delta Q}{Q}.$$

Oftentimes, when a quantity is known with high precision, the relative uncertainty is expressed with its inverse, like a scale factor.

Example 1.3. *The relative uncertainty on the closet width w in example 1.1 is $\frac{0.2}{61.4} \simeq 0.0032 = 0.32\%$, or about 1 part in 300. The relative uncertainty on the fine-structure constant α (example 1.2) is 0.000 000 000 023, or about 1 part in 4.3 billion. This is the most precisely known physical constant.*

Exercise 1.1. *For each quantity, state the measurement interval and the relative uncertainty.*

(a) *mass of a frog: 10.3 g $\pm$ 0.05 g*
(b) *temperature in an oven: 120 °C $\pm$ 5 °C*

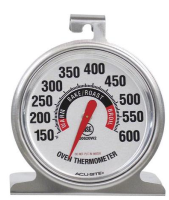
Exercise 1.2. *Transcribe the measurement intervals into a magnitude and uncertainty.*

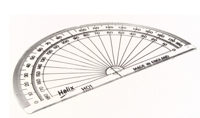
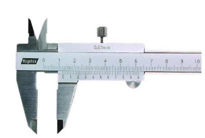
(a) [12.25 kg, 12.35 kg]
(b) [5300 °C, 5350 °C]

Exercise 1.3. *The age of the universe is estimated at 13.8 billion years, with a relative uncertainty of 0.145%.*

(a) *Find the absolute uncertainty and state the measurement interval.*
(b) *Express the relative uncertainty in the form "1 part in . . . ".*

Exercise 1.4. *Estimate or research the absolute and relative uncertainties of these measurement devices. Discuss with your classmates.*

kitchen scale, person scale, thermometer (oven), thermometer (medical), stopwatch, speed meter, protractor, vernier scale

Exercise 1.5. *The side length of a square is 3.25(5) m. Find the measurement interval for its area.*

2 Scientific notation

Very large and very small numbers are cumbersome to write in decimal representation. This is why, especially in science, a more compact form of writing is being used:

Definition. *The* ***scientific notation*** *of a real number x is:*

$$x = \pm m \times 10^n,$$

where m (the ***mantissa****) is a real number in the interval $[1; 10)$ and n (the* ***exponent****) is an integer.*

Example 2.1. *The eighth Mersenne prime is the number*

$$2^{31} - 1 = 2\,147\,483\,647 = 2.147\,483\,647 \times 10^9 \simeq 2.15 \times 10^9.$$

Example 2.2. *The mass of the Earth is*

$$M_\oplus \simeq 5\,972\,200\,000\,000\,000\,000\,000\,000 \ kg = 5.9722 \times 10^{24} \ kg.$$

The absolute uncertainty on this value is 6×10^{20} kg, which can be written

$$M_\oplus = 5.9722(6) \times 10^{24} \ kg = (5.9722 \pm 0.0006) \times 10^{24} \ kg.$$

The scientific notation separates the **order of magnitude** of a quantity from its more detailed sequence of digits. This allows very large numbers to be entered into a pocket calculator (who can only display a dozen or less digits) using the button "EE" = *enter exponent.* For example, in order to enter the number 9.1×10^{17}, we type

`9` `.` `1` `EE` `1` `7` `=`

Very small numbers have negative exponents, following the pattern

$$\begin{aligned}
10^2 &= 100 \\
10^1 &= 10 \\
10^0 &= 1 \\
10^{-1} &= 0.1 \\
10^{-2} &= 0.01 \\
10^{-3} &= 0.001 \\
&\cdots
\end{aligned}$$

Very small numbers are entered into a calculator as follows (example: 8.03×10^{-18}):

`8` `.` `0` `3` `EE` `1` `8` `+/-` `=`

Theorem. *The decimal form of the number 10^n, where n is an integer, consists of a 1*

- ***followed*** *by n zeroes if n is positive,*
- ***preceded*** *by $|n|$ zeroes if n is negative (including the zero in front of the decimal point).*

Example 2.3. *The fine-structure constant and its absolute uncertainty can be written:*

$$\alpha = 7.297\,352\,566\,4 \times 10^{-3} \pm 1.7 \times 10^{-12},$$

and its relative uncertainty is 2.3×10^{-11}.

In the metric system, as it is implemented in the International System of Units (SI for *Système International*), unit names allow prefixes to denote decimal multiples:

prefix name	symbol	factor	short scale (US)	long scale (UK)
deca	da	10^1	ten	ten
hecto	h	10^2	hundred	hundred
kilo	k	10^3	thousand	thousand
mega	M	10^6	million	million
giga	G	10^9	billion	thousand million
tera	T	10^{12}	trillion	billion
peta	P	10^{15}	quadrillion	thousand billion
exa	E	10^{18}	quintillion	trillion
zetta	Z	10^{21}	sextillion	thousand trillion
yotta	Y	10^{24}	septillion	quadrillion

The prefix is chosen so that the magnitude is a number between 1 and 1000. The SI prefixes are used in conjunction with technical units, such as watts (W), liters (l) or bytes (B). For historical reasons, it is uncommon to use them on lengths, times or masses (there are no “gigameters”, “petagrams” or “megayears”).

Here are the prefixes for decimal subdivisions. They can be used on the SI units of any type of quantity.

prefix name	symbol	factor	short scale (US)	long scale (UK)
deci	d	10^{-1}	tenth	tenth
centi	c	10^{-2}	hundredth	hundredth
milli	m	10^{-3}	thousandth	thousandth
micro	μ	10^{-6}	millionth	millionth
nano	n	10^{-9}	billionth	thousand millionth
pico	p	10^{-12}	trillionth	billionth
femto	f	10^{-15}	quadrillionth	thousand billionth
atto	a	10^{-18}	quintillionth	trillionth
zepto	z	10^{-21}	sextillionth	thousand trillionth
yotto	y	10^{-24}	septillionth	quadrillionth

Exercise 2.1.

The eight planets in our solar system have very different masses:

Planet	***Mass***
Mercury	*330 200 000 000 000 000 000 000 kg*
Venus	*4 869 000 000 000 000 000 000 000 kg*
Earth	*5 974 000 000 000 000 000 000 000 kg*
Mars	*641 900 000 000 000 000 000 000 kg*
Jupiter	*1 899 000 000 000 000 000 000 000 000 kg*
Saturn	*568 500 000 000 000 000 000 000 000 kg*
Uranus	*86 830 000 000 000 000 000 000 000 kg*
Neptune	*102 430 000 000 000 000 000 000 000 kg*

(a) *Write the masses in scientific notation.*
(b) *Sort the planets by mass, from largest to smallest.*

Exercise 2.2. *Write in decimal form:*

(a) 9.1×10^2
(b) 4.43×10^9
(c) 1.03×10^{22}
(d) *30 quadrillion 421 billion (short scale)*
(e) 4.5×10^{-5}
(f) 9.091×10^{-1}
(g) -1.23×10^{-2}

Exercise 2.3. *Write in scientific notation:*

(a) 94 800
(b) 329 800 000
(c) 15 530 000 000 000 000

(d) *12 540 billion 300 million (long scale)*
(e) 0.000 000 005922
(f) $\dfrac{1}{5\,\textit{trillion}}$ *(short scale)*

Exercise 2.4. *Write in decimal and scientific form:*

(a) 0.707×10^6
(b) 61×10^{-3}
(c) $(1204 \times 10^7) \times (0.3 \times 10^5)$
(d) 0.43×10^{15}
(e) $(4.55 \times 10^{-13}) : (2 \times 10^{10})$

Exercise 2.5. *Complete the table.*

	decimal form	*scientific form*	*SI unit*
mean distance Earth–Mars	228 000 000 000 m		–
age of the Sun		4.6×10^9 years	–
mass of the Sun ($M_\odot$)		1.989×10^{30} kg	–
heating power of the Sun			384.6 YW
mass of a carbon-12 atom	0.000 000 000 000 000 000 000 019 926 467 1 g		
shortest light pulse		1.7×10^{-17} s	
typical diameter of an atom			100 pm
number of cells in a human body	100 000 000 000 000		–
water content of the Mediterranean		4.3×10^{18} l	
typical capacity of a hard drive			1 TB

3 Significant figures

Definition. *The* ***significant figures*** *of a quantity are the non-zero digits whose place value is greater than the absolute uncertainty. They are numbered from left to right. A quantity known to n significant figures has a relative uncertainty of about 10^{-n}.*

Example 3.1. *A population clock shows the current world population, increasing in "real time" by 2–3 people per second (see e. g. worldometers.info). This purported precision is of course ridiculous. Serious estimates show no more than four significant figures, i. e. the uncertainty is at least $1 : 10^4 = 0.01\%$, or 1 million people:*

$$7\,40\underset{\substack{\uparrow \\ \textit{last significant figure}}}{1}\,255\,094 \simeq 7\,401\,000\,000$$

In a measurement or calculation, the result often contains more digits than are actually significant. In this situation, the quantity is rounded to the last significant digit. The rounded digit is underlined to signal the rounding error, or level of uncertainty.

Example 3.2.

$$\begin{aligned} 6\,365\,964 &\simeq 6\,36\underline{6}\,000 = 6.36\underline{6} \times 10^6 \\ \frac{142}{665} = 0.0\overline{852} &\simeq 0.08\underline{5} = 8.\underline{5} \times 10^{-2} \end{aligned}$$

Figure 3.1: Population clock of the US Census Bureau *(census.gov)*: an example of false precision

Figure 3.2: "Peak XV", a. k. a. Mount Everest

When the digit being rounded to is 0 (**trailing zero**), it is kept (even after a decimal point) and underlined to indicate that it is significant and not the result of rounding.

Example 3.3. *During the Great Trigonometrical Survey of India, led by Surveyor-General Sir Andrew Scott Waugh in 1843–1861, the height of "Peak XV" in the Himalayas was measured to be 29 000 ft ± 1 ft*[1]*. However, the height was reported officially as 29 002 ft in 1850, so as not to give the impression that the trailing zeroes were the result of rounding. In 1865, Peak XV was named after Waugh's predecessor, Sir George Everest. Waugh thus became the first person to put two feet on Mt. Everest.*

When calculating with rounded quantities, the following rules apply:

- The result of an **addition** or **subtraction** is rounded to the **position of the least significant digit** in the most uncertain operand.
- The result of a **multiplication** or **division** is rounded to the **number of significant digits** in the most uncertain operand.

Example 3.4.

(a) $494\,\underline{2}00 + 5\underline{3}\,000 \simeq 54\underline{7}\,000$

(b) $\underline{8}\,000\,000 - 1 \simeq \underline{8}\,000\,000$

(c) $5\,1\underline{0}0 \times 0.000\,339\,7\underline{6} = 5.1\underline{0} \times 10^3 \times 3.397\underline{6} \times 10^{-4} \simeq 17.\underline{3} \times 10^{-1} = 1.7\underline{3}$

[1] 1 foot = 1 ft = 1' = 30.48 cm

Exercise 3.1. *Round these numbers to two significant figures and write the result in scientific notation.*

(a) $3\,517\,428.906$
(b) $45\,690\,973$
(c) $0.\overline{505} \times 10^{-4}$

Exercise 3.2. *Round these numbers to five significant figures and write the result in decimal form and in scientific form.*

(a) $6\,666.666$
(b) $0.\overline{078}$
(c) π^{20}

Exercise 3.3. *Calculate with the appropriate rounding. Write the result in decimal form and scientific notation.*

(a) $82.\underline{7} + 19\underline{3}$
(b) $0.1428\underline{6} \times 7$
(c) $0.12\underline{5} \times 7.5\underline{5} \times 10^{-3} + 0.652\underline{5} \times 10^{-4}$

Exercise 3.4. *The speed of light (in a vacuum) is* $c = 3.0\underline{0} \times 10^8$ *m/s. There are* $365.2\underline{5}$ *days in a year. A light-year is the distance light travels in a year. How long is a light-year? Give your answer in scientific notation with appropriate rounding.*

Exercise 3.5. *The density of water at room temperature is* $\rho = 99\underline{8}$ *g/l. Find the volume of 5 tons of water at room temperature. Give your answer in decimal form with appropriate rounding.*

4 Solutions to the exercises

1.1 (a) $[10.25\,\mathrm{g}, 10.35\,\mathrm{g}]$, 4.9%; (b) $[115\,^\circ\mathrm{C}, 125\,^\circ\mathrm{C}]$, 4.2%
1.2 (a) $12.3\,\mathrm{kg} \pm 0.05\,\mathrm{kg}$; (b) $5325\,^\circ\mathrm{C} \pm 25\,^\circ\mathrm{C}$
1.3 (a) 20 million years, [13.78 billion years, 13.82 billion years]; 1 part in 690
1.5 $[10.24\,\mathrm{m}^2, 10.89\,\mathrm{m}^2]$
2.1

Planet	**Mass**
Jupiter	1.899×10^{27} kg
Saturn	5.685×10^{26} kg
Neptune	1.0243×10^{26} kg
Uranus	8.683×10^{25} kg
Earth	5.974×10^{24} kg
Venus	4.869×10^{24} kg
Mars	6.419×10^{23} kg
Mercury	3.302×10^{23} kg

2.2 (a) 910; (b) 4 430 000 000; (c) 10 300 000 000 000 000 000 000; (d) 30 000 421 000 000 000; (e) 0.000 045; (f) 0.9091; (g) -0.0123
2.3 (a) 9.48×10^4; (b) 3.298×10^8; (c) 1.553×10^{16}; (d) $1.254\,000\,03 \times 10^{16}$; (e) 5.922×10^{-9}; (f) 2×10^{-19}
2.4 (a) $7.07 \times 10^5 = 707\,000$; (b) $6.1 \times 10^{-2} = 0.061$; (c) $3.612 \times 10^{14} = 361\,200\,000\,000\,000$; (d) $4.3 \times 10^{14} = 430\,000\,000\,000\,000$; (e) $9.1 \times 10^{-3} = 0.0091$

2.5

	decimal form	*scientific form*	*SI unit*
mean distance Earth–Mars	228 000 000 000 m	2.28×10^{11} m	–
age of the Sun	4 600 000 000 years	4.6×10^{9} years	–
mass of the Sun ($M_\odot$)	1 989 000 000 000 000 000 000 000 000 000 kg	1.989×10^{30} kg	–
heating power of the Sun	384 600 000 000 000 000 000 000 000 W	3.846×10^{26} W	384.6 YW
mass of a carbon-12 atom	0.000 000 000 000 000 000 000 019 926 467 1 g	$1.992\,646\,71 \times 10^{-21}$ g	19.992 646 71 yg
shortest light pulse	0.000 000 000 000 000 017 s	1.7×10^{-17} s	17 as
typical diameter of an atom	0.000 000 000 1 m	10^{-10} m	100 μm
number of cells in a human body	100 000 000 000 000	10^{14}	–
water content of the Mediterranean	4 300 000 000 000 000 000 l	4.3×10^{18} l	4.3 El
typical capacity of a hard drive	1 000 000 000 000 B	10^{12} B	1 TB

3.1 (a) $3.\underline{5} \times 10^{6}$; (b) $4.\underline{6} \times 10^{7}$; (c) $5.\underline{1} \times 10^{-5}$
3.2 (a) $6\,666.\underline{7} = 6.666\underline{7} \times 10^{3}$; (b) $0.07807\underline{8} = 7.807\underline{8} \times 10^{-2}$; (c) $8.770\underline{0} \times 10^{9}$
3.3 (a) $27\underline{6} = 2.7\underline{6} \times 10^{2}$; (b) $1.000\underline{0} = 1.000\underline{0} \times 10^{0}$; (c) $0.0010\underline{1} = 1.0\underline{1} \times 10^{-3}$
3.4 $9.4\underline{7} \times 10^{15}$ m
3.5 $5.0\underline{1} \times 10^{3}$ l