Real numbers

Contents

1 Completing the number line

The decimal representation of a rational number $\frac{p}{q}$ (p, q integers) is either:

- an integer: $\dfrac{24}{6} = 4$
- a terminating decimal number: $\dfrac{1}{8} = 0.125$
- or a repeating decimal number: $\dfrac{33}{14} = 2.3\overline{571428}$

But of course, it is very easy to think of decimal numbers that fall into neither category, i. e. that have an infinite number of digits that do not repeat:

- $1.01001000100001000001\ldots$
- $0.123456789101112131415\ldots$ (Champernowne constant)
- $0.121232123432123454321\ldots$
- $24.68101214161820\ldots$
- $0.23571113171923\ldots$ (Copeland-Erdős constant)

Definition. *A decimal number that cannot be expressed as a fraction of integers is called an* ***irrational number*** *("not a ratio"). The irrational numbers complement the rationals $\mathbb{Q}$ to form the set of* ***real numbers*** *$\mathbb{R}$.*

The rational numbers, while being packed infinitely close on the number line, do not fill it out. Many important numbers are irrational, such as π, $\sqrt{2}$ and most other roots of natural numbers ($\sqrt{3}, \sqrt{5}, \sqrt[3]{2}, \ldots$), as we will see in the chapter "Powers and roots".

The sets of numbers $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are all part (subsets) of the real numbers $\mathbb{R}$. They stack into each other like Russian dolls (Fig. 1.1), which we write using using the symbol $\subset$ meaning "is a subset of":

- Every natural number is an integer, so $\mathbb{N} \subset \mathbb{Z}$.
- Every integer is rational, so $\mathbb{Z} \subset \mathbb{Q}$.
- Every rational number is real, so $\mathbb{Q} \subset \mathbb{R}$.

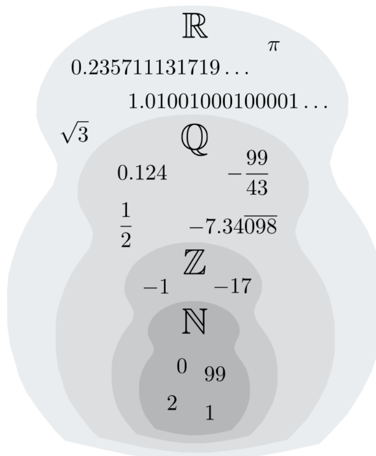


Figure 1.1: The hierarchy of number sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$

Exercise 1.1. *Mark for each number to which number sets it belongs to and to which not. Which numbers are rational, which are irrational?*

	$\in \mathbb{N}$	$\in \mathbb{Z}$	$\in \mathbb{Q}$	$\in \mathbb{R}$	*irrational*
$\frac{355}{113}$					
$0.1491625364964\ldots$					
$0.1234543212345\ldots$					
17					
$-0.\overline{4}$					
$\frac{1.75}{0.25}$					
$23 : 707$					
$\frac{1}{99^2}$					
$\frac{1}{\sqrt{2}}$					

Exercise 1.2. *True or false?*

(a) Every natural number is rational.
(b) A number is either rational or real.
(c) No integer is rational.

(d) *Irrational numbers are not real.*
(e) $\mathbb{N} \subset \mathbb{R}$
(f) $\mathbb{Q} \subset \mathbb{Z}$

Exercise 1.3. *Knowing that* π *is irrational, explain why* $\sqrt{\pi}$ *is also irrational. Start your argument like this:*

> *"If* $\sqrt{\pi}$ *were rational, it could be written as a fraction* $\frac{p}{q}$ *of integer numbers* p *and* q*. But then* $\pi = \dots$*"*

Exercise 1.4. *This exercise shows that the number*

$$e = 2 + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \frac{1}{1 \times 2 \times 3 \times 4} + \dots + \frac{1}{n!} + \dots$$

is irrational. (The notation $n!$*, or "*n *factorial", is a shorthand for the product of the first* n *natural numbers* $1 \times 2 \times 3 \times \dots \times n$*, e. g.* $4! = 1 \times 2 \times 3 \times 4 = 24$*.)*

(a) *Complete the table to get a sequence of increasingly precise approximations of* e*.*

n	$n!$	$\frac{1}{n!}$	$2 + \frac{1}{2!} + \dots + \frac{1}{n!}$
2			
3			
4			
5			
6			
7			
8			

(b) *If* e *were a rational number* $\frac{p}{q}$ $(p, q \in \mathbb{N})$*, multipliying it with* q *(or any multiple of* q*) would turn it into a natural number. Let us therefore multiply it with some factorial* $q!$*. Develop* $q!e$ *into a sum. Which terms of the sum are integers, and which are not?*
(c) *For* $q!e$ *to be a natural number, the non-integer terms must sum to an integer. Show that the sum of the non-integer terms is less than* $\frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3} + \dots$
(d) *Find approximate values of this last sum for* $q = 2$*. Explain why the value of the infinite sum* $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$ *cannot be greater than* 1*.*
(e) *Thus explain why the sum of non-integer terms in* $q!e$ *is less than* 1*, and thus why* e *is irrational.*

2 Subsets of the real numbers

Definition. *A* ***set of numbers*** S *is an infinite, finite or even empty collection of numbers. The numbers contained in* S *are called its* ***elements****. Any number* x *is either contained in the set:* $x \in S$*, or not contained in it:* $x \notin S$*.*

Sets of numbers can be described in several ways:

- by **complete enumeration** (the order does not matter):
  - the set of natural numbers less than 5: $\{0, 1, 2, 3, 4\}$
  - the set of prime numbers between 10 and 30: $\{11, 13, 17, 19, 23, 29\}$
  - the set of integers from -3 to 3: $\{-3, -2, -1, 0, 1, 2, 3\}$
  - the set of integers between -3 and 3: $\{-2, -1, 0, 1, 2\}$
  - the set of primitive fractions whose value is greater than $\frac{1}{\pi}$: $\{1, \frac{1}{2}, \frac{1}{3}\}$
  - the empty set: $\{\} = \emptyset$
  - the set containing just the number 0: $\{0\}$
  - given any real number x, the set containing just x: $\{x\}$
- by **partial enumeration**, followed by an ellipsis:
  - the set of naturals from 0 to 100: $\{0, 1, 2, 3, \ldots, 100\}$
  - the set of inverses of the prime numbers: $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\}$
  - the set of integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} = \{0, 1, -1, 2, -2, \ldots\}$
  - the set of prime numbers: $\mathbb{P} = \{2, 3, 5, 7, 11, \ldots\}$
- by a **base set** and a **condition**:
  - the set of real numbers greater than 2: $\{x \in \mathbb{R} \mid x > 2\}$
  - the set of real numbers whose square is 5: $\{x \in \mathbb{R} \mid x^2 = 5\} = \{\sqrt{5}, -\sqrt{5}\}$
  - the set of rational numbers whose square is 5: $\{x \in \mathbb{Q} \mid x^2 = 5\} = \emptyset$
  - the set of irrational numbers: $\{x \in \mathbb{R} \mid \text{there exist no } p, q \in \mathbb{Z} \text{ such that } x = \frac{p}{q}\} = \{x \in \mathbb{R} \mid \text{for no integer } q \text{ is } qx \in \mathbb{Z}\}$
  - the set of even naturals that can be written as the sum of two primes: $\{x \in \mathbb{N} \mid x \text{ is even and } x = p_1 + p_2 \text{ for some } p_1, p_2 \in \mathbb{P}\}$

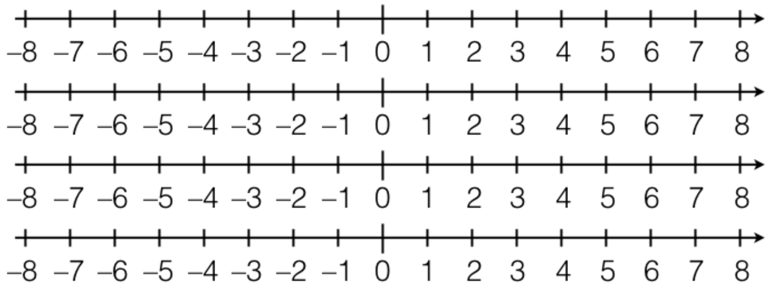


Figure 2.1: The intervals $(-4;3)$, $[-4;3]$, $(-4;3]$ and $[-4;3)$

It is unknown whether the last set contains all even naturals or not (Goldbach conjecture).

Definition. *Given two real numbers $a < b$, then the **open interval** $(a;b)$ is the set of real numbers between a and b:*

$$(a;b) = \{x \in \mathbb{R} \mid a < x < b\}.$$

*The **closed interval** is the set*

$$[a;b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

In analogous fashion we understand the half-open intervals $(a;b]$ and $[a;b)$ (Fig. 2.1).

Definition. *A real number a gives rise to the **open rays***

$$(a;\infty) = \{x \in \mathbb{R} \mid x > a\}$$

and

$$(-\infty;a) = \{x \in \mathbb{R} \mid x < a\},$$

*as well as the analogously defined **closed rays** $[a;\infty)$ and $(-\infty;a]$ (Fig. 2.2)*

Definition. *Sets can be combined in several ways to form new sets. Given two sets A and B, we define:*

*(a) the **intersection** of A and B:*

$$A \cap B = \{x \in \mathbb{R} \mid x \in A \text{ and } x \in B\}$$

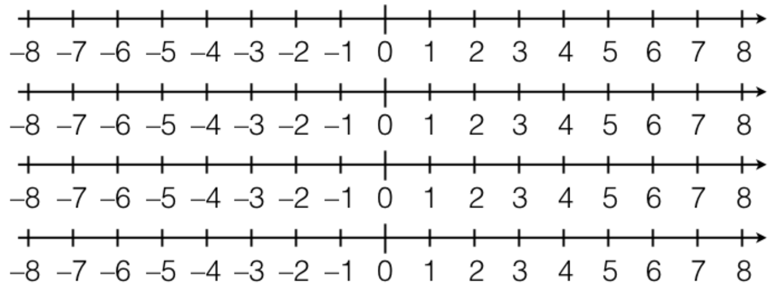


Figure 2.2: The rays $(2;\infty)$, $[2;\infty)$, $(-\infty;2]$ and $(-\infty;2)$

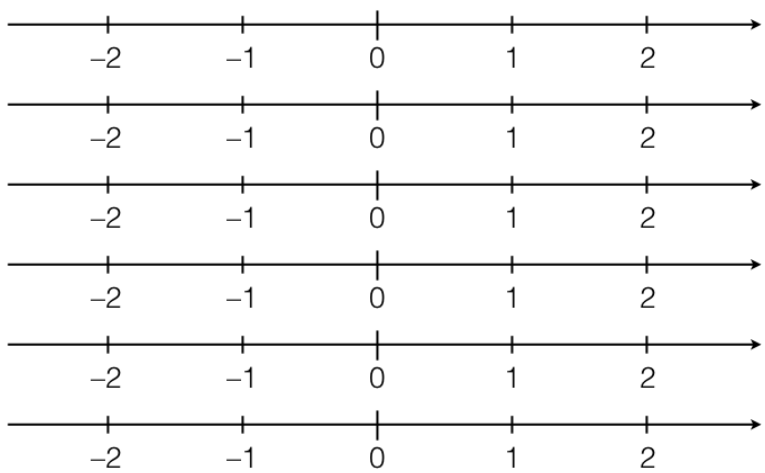


Figure 2.3: Operations on sets (example 2.1)

(b) the ***union*** *of A and B:*

$$A \cup B = \{x \in \mathbb{R} \mid x \in A \text{ or } x \in B \text{ (or both)}\}$$

(c) the ***complement*** *of B in A:*

$$A \setminus B = \{x \in \mathbb{R} \mid x \in A \text{ but } x \notin B\}$$

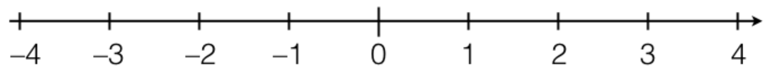
Example 2.1. *Consider the sets $A = \{x \in \mathbb{R} \mid 1 \leq |x| < 2\}$ and $B = [-1;1.5)$. The set A is the union of two separate intervals: $A = (-2;-1] \cup [1;2)$, and B (as any interval) is the intersection of two rays: $B = (\infty;1.5) \cap [-1;\infty)$. The union, intersection and complements of both sets are (Fig. 2.3):*

(a) $A \cup B = (-2; 2)$
(b) $A \cap B = \{-1\} \cup [1; 1.5)$
(c) $A \setminus B = [1.5; 2)$
(d) $B \setminus A = (-1; 1)$

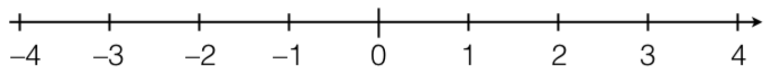
The set of irrational numbers can thus be written as the complement of $\mathbb{Q}$ in $\mathbb{R}$, i. e. $\mathbb{R} \setminus \mathbb{Q}$.

Exercise 2.1. *Represent the sets on the number line.*

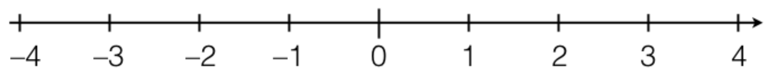
(a) $(-3; 3)$



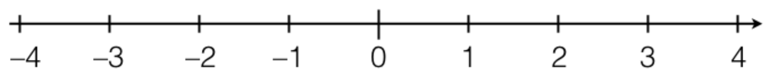
(b) $[-3; 3]$



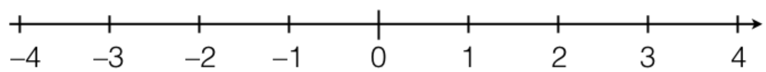
(c) $\{-3, 3\}$



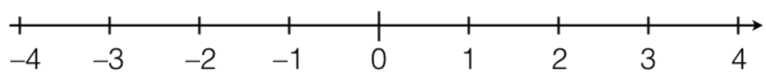
(d) $\{-3.3\}$



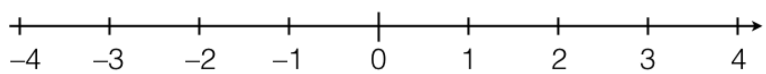
(e) $(1; 3) \cup [-2; 0)$



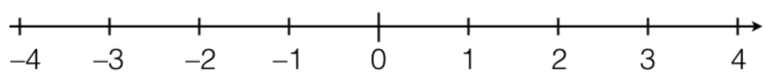
(f) $\mathbb{N} \cap (-\infty; 4)$



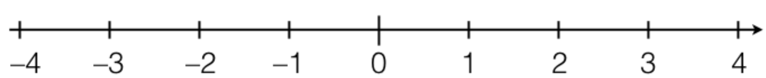
(g) $[-4; -1] \cap (-1; 2)$



(h) $\mathbb{R} \setminus \{\sqrt{2}, -\sqrt{2}\}$

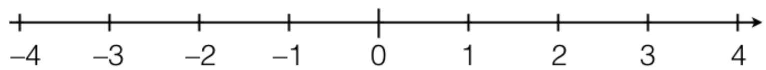


(i) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

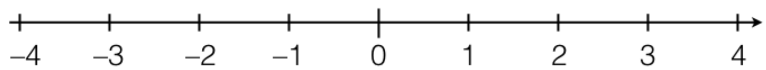


Exercise 2.2. *Let* $A = \{x \in \mathbb{R} \mid 2x \text{ is natural}\}$ *and* $B = \{x \in \mathbb{R} \mid x \leq 2\}$. *Represent on the number line and write in set or interval notation:*

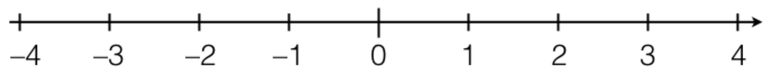
(a) A



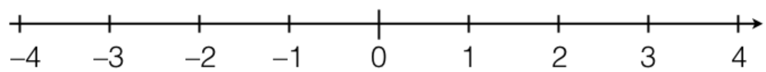
(b) B



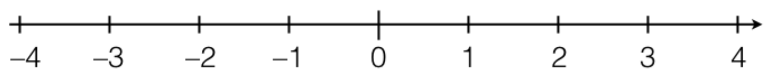
(c) $A \cup B$



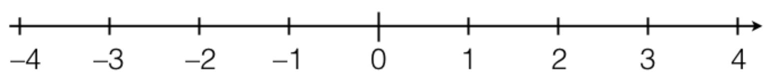
(d) $A \cap B$



(e) $A \setminus B$



(f) $B \setminus A$



3 Solutions to the exercises

1.1

	$\in \mathbb{N}$	$\in \mathbb{Z}$	$\in \mathbb{Q}$	$\in \mathbb{R}$	irrational
$\dfrac{355}{113}$	✗	✗	✓	✓	✗
$0.1491625364964\ldots$	✗	✗	✗	✓	✓
$0.1234543212345\ldots$	✗	✗	✓	✓	✗
17	✓	✓	✓	✓	✗
$-0.\overline{4}$	✗	✗	✓	✓	✗
$\dfrac{1.75}{0.25}$	✓	✓	✓	✓	✗
$23 : 707$	✗	✗	✓	✓	✗
$\dfrac{1}{99^2}$	✗	✗	✓	✓	✗
$\dfrac{1}{\sqrt{2}}$	✗	✗	✗	✓	✓

1.2 (a) true; (b) false; (c) false; (d) false; (e) true; (f) false

1.4

n	$n!$	$\frac{1}{n!}$	$2+\frac{1}{2!}+\cdots+\frac{1}{n!}$
2	2	0.5	2.5
3	6	$0.1\overline{6}$	$2.\overline{6}$
4	24	$0.041\overline{6}$	$2.708\overline{3}$
5	120	$0.008\overline{3}$	$2.71\overline{6}$
6	720	$0.0013\overline{8}$	$2.7180\overline{5}$
7	5040	$\simeq 0.0001984127$	$\simeq 2.7182539683$
8	40 320	$\simeq 0.0000248016$	$\simeq 2.7182787698$

2.1 (a) $A = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$; (b) $B = (-\infty; 2]$; (c) $A \cup B = (-\infty; 2] \cup \{\frac{5}{2}, 3, \frac{7}{2}, \ldots\}$; (d) $A \cap B = \{0, \frac{1}{2}, 1\frac{3}{2}, 2\}$; (e) $A \setminus B = \{\frac{5}{2}, 3, \frac{7}{2}, \ldots\}$; (f) $B \setminus A = (-\infty; 2) \setminus \{0, \frac{1}{2}, 1, \frac{3}{2}\} = (-\infty; 0) \cup (0; \frac{1}{2}) \cup (\frac{1}{2}; 1) \cup (1; \frac{3}{2}) \cup (\frac{3}{2}; 2)$