Powers and roots

Contents

1 Square roots

Definition. *Given a real number a, then the* ***square root*** *of a, written $\sqrt{a}$, is the* ***positive*** *solution (if it exists) of the equation*

$$x^2 = a,$$

i. e. it is

$$(\sqrt{a})^2 = a \text{ and } \sqrt{a} \geq 0.$$

The expression under a square root is called the ***radicand****.*

An immediate result of this definition is that we can only take the square root of non-negative radicands:

Theorem. *Given the equation $x^2 = a$ for some $a \in \mathbb{R}$.*

(a) If $a < 0$, the equation has no solution.
(b) If $a = 0$, the equation has the unique solution $x = 0$.
(c) If $a > 0$, the equation has two solutions: $x = \sqrt{a}$ and $x = -\sqrt{a}$.

The square root is thus the inverse operation of squaring, but one must be careful with the signs:

Theorem. *For any $a \in \mathbb{R}$:*

(a) $(\sqrt{a})^2 = a$ (if $a \geq 0$, otherwise the expression is not defined)
(b) $\sqrt{a^2} = |a|$

So for example, $\sqrt{(-5)^2} = \sqrt{25} = 5$, and not -5!

Example 1.1. *The area A of a circle is, as an expression in its radius r, $A = \pi r^2$. This formula can be solved for r, so that the radius can be found from the area:*

$$r^2 = \frac{A}{\pi} \Rightarrow r = \sqrt{\frac{A}{\pi}}.$$

For example, a bucket of paint that lasts for $A = 2\underline{0}0 \text{ m}^2$ can be used to paint a filled circle of radius $r = \sqrt{\frac{200}{\pi}} \simeq 8.\underline{0}$ m.

The square root of a rational number is found via its prime factorization:

Example 1.2.

(a) $\sqrt{7056} = \sqrt{2^4 \times 3^2 \times 7^2} = \sqrt{(2^2 \times 3 \times 7)^2} = 2^2 \times 3 \times 7 = 84$

(b) $\sqrt{\dfrac{3136}{20\,449}} = \sqrt{\dfrac{2^6 \times 7^2}{11^2 \times 13^2}} = \sqrt{\left(\dfrac{2^3 \times 7}{11 \times 13}\right)^2} = \sqrt{\left(\dfrac{56}{143}\right)^2} = \dfrac{56}{143}$

But this method only works if the prime factorization contains just even powers of primes, so that the radicand can be factored into a square. In particular:

Theorem. *The square root of a natural number is either natural itself or irrational. In particular, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, … are irrational numbers.*

Proof. The square $\frac{a^2}{b^2}$ of a rational number $\frac{a}{b}$ (a, $b \in \mathbb{Z}$), when factored into primes, only contains even exponents. Therefore, if a rational contains unpaired prime factors, such as $1224 = 2^3 \times 3^2 \times 17$ (contains unpaired factors 2 and 17), it cannot be the square of a rational number. Its square root is therefore irrational. This applies in particular to natural numbers that contain an unpaired prime factor, i. e. the non-squares. $\square$

Exercise 1.1. *Find all solutions of each equation. Which solutions are irrational? Give them in exact and approximate form.*

(a) $x^2 = 36$
(b) $x^2 + 7 = 0$
(c) $x^2 = 99$
(d) $x^4 = \pi$

Exercise 1.2. *Claire explains: "I cannot tell you which number x I am thinking of, but I can give you a hint: $\sqrt{-x} = 5$." Can you find the number x or is Claire mistaken? Explain.*

Exercise 1.3. *For which x (subset of $\mathbb{R}$) is*

(a) $\sqrt{x} < x$?
(b) $\sqrt{x} > x$?
(c) $x^2 \leq x$?
(d) $x^2 > x$?
(e) $\sqrt{x} = x^2$?

Exercise 1.4.

(a) Imagine a circle with radius r transformed into a square of equal area. Find the square's sidelength as an expression in r.
(b) Imagine a square of sidelength s transformed into a circle of equal area. Find the circle's diameter as an expression in s.

Exercise 1.5. *Calculate by hand. Factorizations can be done with the help of a calculator.*

(a) $\sqrt{9801}$

(b) $\sqrt{73\,984}$

(c) $\sqrt{\dfrac{324}{1225}}$

(d) $\sqrt{\dfrac{4608}{4802}}$

(e) $\sqrt{1.36\overline{1}}$

(f) $\sqrt{\sqrt{0.0625}}$

Exercise 1.6. *Simplify:*

$$\sqrt{\sqrt{\sqrt{\frac{1}{4}}-\frac{1}{4}}-\frac{1}{4}}-\frac{1}{4}$$

2 Radical expressions

Definition. *A **radical expression** is an expression containing at least one (square) root, in addition to the four base operations $+$, $-$, $\times$ and $:$ as well as powers with natural exponents.*

While it is possible to nest square roots into each other, e. g. $\sqrt{\sqrt{a}+\sqrt{b}}$, we will rarely encounter such expressions. This section focuses on how to simplify **simple**, i. e. unnested radical expressions.

As a reminder, an expression is called **arithmetic** if it contains only numbers and no variables. An expression with variables is called **algebraic**.

Theorem. *For any $a, b \in \mathbb{R}_0^+$:*

(a) $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

(b) $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$

(c) $\sqrt{\dfrac{1}{b}} = \dfrac{1}{\sqrt{b}}$

(d) $\sqrt{a^n} = \left(\sqrt{a}\right)^n$ *($= a^{n:2}$ if n is even)*

Proof. All identities are proven by taking the square of the right-hand side and showing that it equals the radicand of the left-hand side:

(a) $\left(\sqrt{a} \times \sqrt{b}\right)^2 = \left(\sqrt{a}\right)^2 \times \left(\sqrt{b}\right)^2 = ab$

(b) $\left(\dfrac{\sqrt{a}}{\sqrt{b}}\right)^2 = \dfrac{\left(\sqrt{a}\right)^2}{\left(\sqrt{b}\right)^2} = \dfrac{a}{b}$

(c) is a special case of (b), since $\sqrt{1} = 1$

(d) is the straightforward generalization of (a) with the n-fold product $a^n = a \times a \times \ldots \times a$. For the second identity, note that $(a^{n:2})^2 = a^{n:2\times 2} = a^n$.

□

Example 2.1.

(a) $\sqrt{20} \times \sqrt{45} = \sqrt{20 \times 45} = \sqrt{900} = 30$

(b) $\left(\sqrt{2}\right)^{10} = \sqrt{2^{10}} = 2^{10:2} = 2^5 = 32$

However, it is in general

$$\sqrt{a \pm b} \neq \sqrt{a} \pm \sqrt{b} \quad !$$

These algebraic laws can be used to simplify expressions containing square roots. We start with purely arithmetic expressions, namely, square roots of rational numbers. They can be simplified by factoring out as many rational factors as possible. This way, the square root of any rational number can be written in the form $\frac{p}{q}\sqrt{n}$, with natural numbers p, q, n and n is as small as possible. This is called **reducing** a square root.

Example 2.2.

(a) $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$

(b) $\sqrt{405} = \sqrt{3^4 \times 5} = \sqrt{3^4} \times \sqrt{5} = 9\sqrt{5}$

(c) $\sqrt{58\,080} = \sqrt{2^5 \times 3 \times 5 \times 11^2} = \sqrt{(2^2 \times 11)^2} \times \sqrt{2 \times 3 \times 5} = 44\sqrt{30}$

(d) $\sqrt{\frac{3}{2}} = \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{\sqrt{4}} = \frac{1}{2}\sqrt{6}$

(e) $\sqrt{\frac{32}{343}} = \sqrt{\frac{2^5}{7^3}} = \sqrt{\left(\frac{2^2}{7}\right)^2 \times \frac{2}{7}} = \frac{4}{7}\sqrt{\frac{2}{7}} = \frac{4}{7}\sqrt{\frac{14}{49}} = \frac{4}{7}\frac{\sqrt{14}}{\sqrt{49}} = \frac{4}{49}\sqrt{14}$

Square roots of polynomials are simplified in the same way. The goal is always to make the radicands as simple as possible, by extracting square factors out of the root:

Example 2.3.

(a) $\sqrt{a^7} = \sqrt{a^6 \times a} = \sqrt{a^6} \times \sqrt{a} = a^3\sqrt{a}$ *(Note: $\sqrt{a}$ exists because $a \geq 0$, since $a^7 \geq 0$)*

(b) $\sqrt{27x^9y^4} = \sqrt{3^2 \times 3 \times x^8 \times x \times y^4} = \sqrt{(3x^4y^2)^2 \times 3x} = 3x^4y^2\sqrt{3x}$ *(why does $\sqrt{3x}$ exist?)*

(c) $\sqrt{(3x)^2 + (4x)^2} = \sqrt{9x^2 + 16x^2} = \sqrt{25x^2} = \sqrt{25} \times \sqrt{x^2} = 5|x|$

(d) $\sqrt{9a^2b^2 + 36b^2} = \sqrt{9b^2(a^2+4)} = \sqrt{(3b)^2} \times \sqrt{a^2+4} = 3|b|\sqrt{a^2+4}$

(e) $\sqrt{a^2 - 2ab + b^2} = \sqrt{(a-b)^2} = |a-b|$

Finally, let's look at fractional expressions with roots in the denominator. First the purely arithmetic expressions: In addition to making the radicands as small as possible, the denominators must be free of square roots.

Definition. *The **normal form** of a simple (unnested) arithmetic radical expression is the sum of expressions of the form $\frac{p}{q}\sqrt{n}$, where $\frac{p}{q} \in \mathbb{Q}$ is irreducible and $n \in \mathbb{N}$ is as small as possible.*

The normal form can always be attained by expanding:

Example 2.4. $\dfrac{2+\sqrt{2}}{\sqrt{3}} = \dfrac{\left(2+\sqrt{2}\right)\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}} = \dfrac{2\sqrt{3}+\sqrt{6}}{3} = \dfrac{2}{3}\sqrt{3}+\dfrac{1}{3}\sqrt{6}$

Sums of square roots can be eliminated in a denominator by making use of the third binomial formula:

Example 2.5. $\dfrac{1}{\sqrt{2}+\sqrt{3}} = \dfrac{\sqrt{2}-\sqrt{3}}{\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{2}-\sqrt{3}\right)} = \dfrac{\sqrt{2}-\sqrt{3}}{\left(\sqrt{2}\right)^2-\left(\sqrt{3}\right)^2} = \dfrac{\sqrt{2}-\sqrt{3}}{-1} =$
$\sqrt{3}-\sqrt{2}$

For algebraic expressions combining fractions and square roots, the same goals apply and are reached in the same fashion:

Example 2.6.

(a) $\dfrac{\sqrt{6b^4}}{\sqrt{10a^3}} = \dfrac{b^2\sqrt{6}}{a\sqrt{10a}} = \dfrac{b^2\sqrt{6}\times\sqrt{10a}}{a\times 10a} = \dfrac{\sqrt{60}}{10}\times\dfrac{b^2\sqrt{a}}{a^2} = \dfrac{\sqrt{15}}{5}\times\dfrac{b^2\sqrt{a}}{a^2} = \dfrac{b^2}{5a^2}\sqrt{15a}$

(b) $\dfrac{\sqrt{a}-\sqrt{b}}{3\sqrt{a}-\sqrt{b}} = \dfrac{\left(\sqrt{a}-\sqrt{b}\right)\left(3\sqrt{a}+\sqrt{b}\right)}{\left(3\sqrt{a}-\sqrt{b}\right)\left(3\sqrt{a}+\sqrt{b}\right)} = \dfrac{3a-3\sqrt{ab}+\sqrt{ab}-b}{9a-b} = \dfrac{3a-b-2\sqrt{ab}}{9a-b}$

Exercise 2.1. *Find, for each square root, its reduced form and its decimal approximation to 4 significant digits.*

(a) $\sqrt{32}$
(b) $\sqrt{540}$
(c) $\sqrt{0.04}$
(d) $\sqrt{0.004}$
(e) $\sqrt{10^7}$
(f) $\sqrt{8.1\times 10^{99}}$
(g) $\sqrt{2^{47}+2^{47}}$

Exercise 2.2. *Simplify.*

(a) $\sqrt{4p^2q^6}$
(b) $\sqrt{18k^2}$
(c) $\sqrt{(mx)^2+(my)^2}$
(d) $\sqrt{4u^2-4uv+v^2}$
(e) $\sqrt{w^3-w^2}$
(f) $\sqrt{10n^2-40n+40}$
(g) $\sqrt{2r}\sqrt{3s}\sqrt{6rs}$

Exercise 2.3. *Simplify.*

(a) $\sqrt{\dfrac{c^3 d}{cd^3}}$

(b) $\dfrac{\sqrt{12r^4st^2}}{\sqrt{27s^5t^2}}$

(c) $\sqrt{\dfrac{25z^4}{z^4+10z^2+25}}$

(d) $\sqrt{\dfrac{ab}{cd}\dfrac{\sqrt{ac}}{\sqrt{bd}}}$

Exercise 2.4. *Simplify into normal form.*

(a) $\sqrt{\dfrac{5}{33}}\sqrt{\dfrac{11}{15}}$

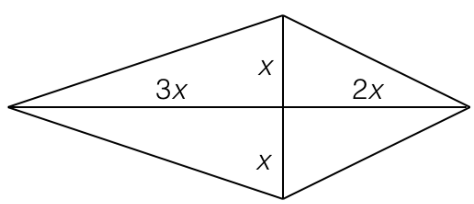
(b) $\sqrt{5}(\sqrt{20}+\sqrt{5})$

(c) $\sqrt{6}\left(\sqrt{\dfrac{1}{2}}-\dfrac{1}{\sqrt{3}}\right)$

(d) $\sqrt{2}-\sqrt{3}+\sqrt{8}-\sqrt{12}-\sqrt{18}+\sqrt{27}$

(e) $\sqrt{3\sqrt{2}}\sqrt{6\sqrt{2}}$

Exercise 2.5. *Caitlin wants to build a kite with the given proportions. The border of the kite will be adorned by a yellow band, of which she has the length ℓ. Find x as an expression in ℓ. (Hint: it will be of the form* constant $\times$ ℓ*, why?)*



Exercise 2.6. *The diagonal of a square is 1 unit longer than its sides. Find the sidelength (exact expression in normal form).*

Exercise 2.7. *Simplify into normal form.*

(a) $\dfrac{\sqrt{3}}{2-\sqrt{6}}$

(b) $\dfrac{\sqrt{15}+\sqrt{10}}{\sqrt{15}-\sqrt{10}}$

(c) $\dfrac{10-7\sqrt{2}}{2\sqrt{3}-\sqrt{6}}$

(d) $\dfrac{\sqrt{r}-2\sqrt{s}}{2\sqrt{r}-\sqrt{s}}$

Exercise 2.8. *Two positive real numbers $a < b$ are said to be in the* ***golden ratio*** *if $b : a = (a+b) : b$. It can be shown (next year!) that the golden ratio is $\varphi = \frac{b}{a} = \frac{\sqrt{5}+1}{2}$.*

(a) *Show that the proportion $\frac{b}{a} = \frac{a+b}{b}$ holds indeed.*
(b) *Show that $\varphi^2 = \varphi + 1$.*
(c) *Show that $\frac{1}{\varphi} = \varphi - 1$.*

Exercise 2.9.

(a) *Expand and simplify $\left(\sqrt{x} - \sqrt{y}\right)^2$.*
(b) *Show that for any two positive real numbers x and y, the arithmetic mean $(\frac{x+y}{2})$ is always greater than or equal their geometric mean $(\sqrt{xy})$.*

Exercise 2.10. *Show that if two numbers $a, b \in \mathbb{R}_0^+$ satisfy $\sqrt{\frac{a+b}{2}} = \frac{\sqrt{a}+\sqrt{b}}{2}$, then they must be equal: $a = b$. Hint: Show from the equation that $\left(\sqrt{a} - \sqrt{b}\right)^2 = 0$.*