Equations and inequalities

Contents

1 General notions

Definition. *An **equation** is a type of assertion about the variables it contains. For some values of the variables, this assertion will be true, for others it will be false, or the values cannot even be inserted.*

*The two expressions on either side of the equal sign are called the **left-hand side** and **right-hand side** (LHS and RHS for short).*

Example 1.1. *The equation*

$$x^2 - \sqrt{y-2} \quad = \quad 5 + \frac{6}{x}$$

is true for the insertion $(x, y) = (3, 6)$, but e. g. false for the insertion $(x, y) = (1, 2)$. The value $x = 0$ cannot be inserted, nor can any y-value less than 2.

1.1 Set of solutions

Definition. *If an equation contains only one variable x, the set of values for x that render the equation true is called the **set of solutions** S of the equation. We say that a solution **satisfies** the equation. The set of solutions may contain one, two, many, infinitely many solutions or no solution at all.*

Example 1.2. *Some examples of equations and their sets of solutions.*

(a) $2x + 3 = -5$ *has a unique solution:* $x = -4$, *so* $S = \{-4\}$
(b) $x^2 = 4$ *has two solutions:* $x = 2$ *and* $x = -2$, *so* $S = \{2; -2\}$
(c) $x(x-3)(x+4) = 0$ *has three solutions:* $x = 0$, $x = 3$ *and* $x = -4$ *(check by insertion), so* $S = \{0; 3; -4\}$ *(why are there no other solutions?)*
(d) $5(x - 3) = 5x - 15$ *has as its set of solutions* $S = \mathbb{R}$
(e) $x^2 + 1 = 0$ *has no solutions (because a square cannot be negative), so* $S = \{\}$ *(the empty set)*

If an equation contains more than one variable, it can be interpreted as an equation in any single one of its variables. This variable is then called the **unknown**, the others are called **parameters**. Solving the equation for the chosen unknown consists in trying to isolate the unknown on the LHS. The RHS is then an expression in the parameters.

The choice of parameters determines not only the values of the solutions, but also how many solutions an equation has.

Example 1.3. *The equation from example 1.1 can be solved for the unknown y:*

$$\begin{aligned}
\sqrt{y-2} &= x^2 - 5 - \frac{6}{x} && \textit{squaring both sides} \\
y - 2 &= \left(x^2 - 5 - \frac{6}{x}\right)^2 && +2 \\
y &= \left(x^2 - 5 - \frac{6}{x}\right)^2 + 2
\end{aligned}$$

Inserting the parameter value $x = 3$ yields the y-value 6, the solution mentioned above. However, inserting $x = 1$ yields $y = 102$, but these two values do not satisfy the original equation ($-9 \neq 11$)! What has gone wrong? We will find out later on.

In general, an equation is solved by rearranging it step by step into simpler, **equivalent equations**. That means they still have the same set of solutions than the original equation. But the previous example shows that even seemingly innocuous steps, such as squaring both sides of an equation, can introduce false (**extraneous**) solutions. Likewise, it is possible to lose valid solutions if one is not careful.

We will investigate this in depth in later sections. As of now, stick to the following steps which preserve the set of solutions in any case:

Theorem. *An equation retains its set of solutions under the following transformations:*

- *simplifying either side into an equivalent expression*
- *exchanging the LHS and RHS*
- *adding or subtracting an expression on both sides*
- *multiplying both sides with a* ***nonzero constant***
- *dividing both sides by a* ***nonzero constant***

1.2 Domain

Sometimes, a solution to an equation can be found, but it is unusable for the underlying problem:

Example 1.4. *Find two consecutive square numbers whose difference is 100.*

Let's write the first square number as n^2, where $n \in \mathbb{N}$. The next square number is $(n+1)^2$, so the equation to solve is

$$(n+1)^2 - n^2 = 100.$$

Expanding the LHS, we get:

$$\begin{array}{rcll} n^2+2n+1-n^2 & = & 100 & \textit{simplify LHS} \\ 2n+1 & = & 100 & -1 \\ 2n & = & 99 & :2 \\ n & = & \dfrac{99}{2} = 49.5 & \end{array}$$

This number indeed solves the equation, but we were looking for a natural number! A solution has been found for the equation, but not for the original problem, since it asked for a natural number. The equation $(n+1)^2 - n^2 = 100$*, as an equation in* $\mathbb{N}$*, has no solutions:* $S = \{\}$*.*

Definition. *The* ***domain*** D *of an equation is the subset of* $\mathbb{R}$ *of permissible solution values. It can be given together with the equation or follows from the underlying problem.*

If no domain is given for a standalone equation, we understand D *to consist of all real numbers that can be arithmetically inserted into the equation (**implicit domain**).*

The set of solutions is therefore more precisely the set of all values x **in the domain** that satisfy the equation, in set notation:

$$S = \{x \in D \mid x \text{ satisfies the equation}\}.$$

Example 1.5.

(a) *A rectangle is 1.5 times as long as it is wide, and its perimeter is 40 m. Calling the width* w*, this amounts to solving the equation* $2(w+1.5w) = 40$*. As the width can only be a positive number, the solutions to this equation are sought in the domain of positive reals* $D = \mathbb{R}^+ = (0; \infty)$*.*

(b) *The Pythagoreans, a school of mathematicians in ancient Greece, thought that any geometric length can be expressed as the ratio between two natural numbers. In other words, they knew only about the rationals* $\mathbb{Q}$*. However, when they tried to find the diagonal of a unit square, i. e. a solution to the equation*

$$x^2 = 2 \textit{ with domain } D = \mathbb{Q},$$

they found no solutions: $S = \{\}$ *(Hippasus, 5th century BCE)! Only in the real numbers,* $D = \mathbb{R}$*, do we find the irrational solutions:* $S = \{\sqrt{2}; -\sqrt{2}\}$*.*

(c) *When given the task to solve the equation*

$$\frac{x}{x-4} = 9 - x,$$

it is implicitly understood that only values other than 4 can be considered, because $x = 4$ is the only value that cannot be inserted. The domain is therefore $D = \mathbb{R} \setminus \{4\}$.

Sometimes, when solving an equation, we find solutions that are actually not in the domain and thus have to be discarded: such false solutions are called **extraneous**.

Example 1.6. *The equation*

$$\frac{x^2 - 3x}{x - 3} \quad = \quad -3,$$

solved naively, yields two solutions:

$$\begin{aligned} \frac{x^2 - 3x}{x - 3} &= -3 && \times (x-3) \\ x^2 - 3x &= -3(x-3) && \textit{expand RHS} \\ x^2 - 3x &= -3x + 9 && + 3x \\ x^2 &= 9 && \sqrt{\ } \\ x = 3 \quad &\textit{or} \quad x = -3. \end{aligned}$$

However, the value $x = 3$ cannot be inserted into the original equation (division by zero)! When we look at the equation more carefully, we see that only values for x other than 3 can be inserted. The equation's domain is thus $D = \mathbb{R} \setminus \{3\}$, and the set of solutions are those values in D that satisfy it: $S = \{-3\}$. The value $x = 3$ is an extraneous solution.

Exercise 1.1. *Consider the equation $(x + 3)^2 = 54 - (x - 3)^2$.*

(a) *Check that $3\sqrt{2}$ is a solution to the equation.*
(b) *Myke thus writes $S = \{3\sqrt{2}\}$. Is this correct?*

Exercise 1.2. *For each equation, find the set of solutions.*

(a) $\dfrac{4x}{5} - 1 = \dfrac{9}{10}$
(b) $2(1 - s) = \dfrac{1 + 2s}{2}$
(c) $x^2 = 9$
(d) $\dfrac{24}{y} = -\dfrac{y}{54}$

Exercise 1.3. *Find a natural number n such that the squares of n and $n + 10$ are 20 apart, or show that there can be no such number.*

Exercise 1.4. *State the implicit domain of these equations. No solving required!*

(a) $1 : x = 4$
(b) $x^2 = 9$
(c) $\dfrac{3x}{x+3} = 5 - x$
(d) $\sqrt{x} = -9$
(e) $\sqrt{x-2} = \dfrac{1}{x-2}$
(f) $\dfrac{x}{x} = 1$

Exercise 1.5.

$$\begin{array}{rcll}
\dfrac{x}{x-1} + \dfrac{1}{x} & = & 1 + \dfrac{1}{x-1} & \times\, x \\
\dfrac{x^2}{x-1} + 1 & = & x + \dfrac{x}{x-1} & \times\,(x-1) \\
x^2 + (x-1) & = & x(x-1) + x & \textit{simplify} \\
x^2 + x - 1 & = & x^2 & -\,x^2 + 1 \\
x & = & 1 & \\
\Rightarrow S & = & \{1\} &
\end{array}$$

*This solution is **wrong**. Explain why.*

2 Linear equations

Definition. *An expression is called **linear** in a variable x if it is a polynomial of degree at most 1 in x. A linear expression is thus of the form*

$$Ax + B,$$

*where A and B are **constants** (expressions not depending on the variable x), or it can be simplified into an expression of that form.*

An equation is linear in its unknown x if both the LHS and the RHS are linear expressions in x.

Example 2.1.

(a) These expressions are all linear in x:

	A	B
$4x$	4	0
$3 - x$	-1	3
$\frac{2x-6}{7}$	$\frac{2}{7}$	$-\frac{6}{7}$
5	0	5
$\frac{ax-1}{a^2-1}$	$\frac{a}{a^2-1}$	$-\frac{1}{a^2-1}$
$rt^2 - xy + 2x$	$2 - y$	rt^2

*(b) These expressions are all **not** linear in x:*

$$x^2, \quad \sqrt{x-1}, \quad |x|, \quad \frac{b}{x}, \quad 1 - sx^3, \quad \frac{ax^2-1}{x^2-1}$$

Example 2.2. *These equations are linear in x:*

(a) $\dfrac{2-x}{3} = -\dfrac{8}{9}\left(\dfrac{x}{4} - 2\right)$
(b) $x\sqrt{y} = 2$
(c) $ax + by = c$

These equations are not linear in x:

(d) $x(x-1) = 5$
(e) $\dfrac{y}{x+1} = 0$
(f) $ax^2 + bx + c = 0$

When both sides of a linear equation in x are expanded, they are sums of expressions that are either constants or constant multiples of x. Those containing x are collected on the LHS and the constants on the RHS.

Theorem. *Any linear equation can be simplified into the form*

$$Ax = B,$$

where x is the unknown and A and B are constants. Then:

(a) If $A \neq 0$, there is a unique solution: $S = \{\frac{B}{A}\}$.
(b) If $A = 0$ but $B \neq 0$, there are no solutions: $S = \{\}$.
(c) If both A and B are 0, any number in the domain solves the equation: $S = D$.

Proof.

(a) This is the definition of the division $\frac{B}{A}$.
(b) No x can make $0x = B$ if $B \neq 0$.
(c) Any x makes $0x = B$ if $B = 0$.

□

Example 2.3.

$$\begin{array}{rcll}
\frac{2-x}{3} & = & \frac{5x}{9} - \frac{8}{9}\left(\frac{x}{4} - 2\right) & D = \mathbb{R},\ \textit{expand both sides} \\
\frac{2}{3} - \frac{x}{3} & = & \frac{5x}{9} - \frac{2}{9}x + \frac{16}{9} & \textit{simplify RHS} \\
\frac{2}{3} - \frac{x}{3} & = & \frac{x}{3} + \frac{16}{9} & -\frac{x}{3} - \frac{2}{3} \\
-\frac{x}{3} - \frac{x}{3} & = & \frac{16}{9} - \frac{2}{3} & \textit{simplify both sides} \\
-\frac{2}{3}x & = & \frac{10}{9} & \times 9 \\
-6x & = & 10 &
\end{array}$$

Hence $A = -6$ and $B = 10$, so the equation has the unique solution $x = \frac{B}{A} = -\frac{5}{3}$. We write $S = \{-\frac{5}{3}\}$.

Example 2.4.

$$\begin{array}{rcll}
1-(1-(1-x)) & = & -(x-1) & D = \mathbb{R},\ \textit{expand and simplify} \\
1-x & = & -x+1 & +x-1 \\
0 & = & 0 &
\end{array}$$

Now $A = B = 0$. Any value of x will make $0 = 0$ true, so $S = D = \mathbb{R}$. This shows that both sides of the original equation are actually equivalent expressions.

Fractions in linear equations can be eliminated early on by multiplying with their denominators, as they are simple constants:

Example 2.5.

$$\begin{aligned}
-2x + \frac{10x+2}{6} &= \frac{2-x}{3} + 1 && D = \mathbb{R}, \times 6 \\
-12x + (10x+2) &= 2(2-x) + 6 && \textit{simplify both sides} \\
-2x + 2 &= 10 - 2x && + 2x - 2 \\
0 &= 8
\end{aligned}$$

The LHS completely loses x upon simplification, so $A = 0$ and $B = 8$. No value of x can make $8 = 0$, so no number can satisfy the original equation: $S = \{\}$.

An equation may initially contain nonlinear expressions that disappear upon simplification.

Example 2.6.

$$\begin{aligned}
(x-3)^2 - 3x^2 &= 2(1-x^2) && D = \mathbb{R}, \ \textit{expand} \\
x^2 - 6x + 9 - 3x^2 &= 2 - 2x^2 && \textit{simplify} \\
-2x^2 - 6x + 9 &= 2 - 2x^2 && + 2x^2 - 9 \\
6x &= -7 && A = 6, \ \ B = -7 \\
x &= -\frac{7}{6}
\end{aligned}$$

So $S = \{-\frac{7}{6}\}$.

2.1 Linear equations with parameters

Now let's take a look at linear equations with a parameter. The value of the parameter determines A and B, and therefore the number of solutions:

Example 2.7.

$$\begin{aligned}
4px - x(p-1) &= x - 6p^2 && D = \mathbb{R}, \textit{expand and simplify} \\
3px + x &= x - 6p^2 && - x \\
3px &= -6p^2
\end{aligned}$$

Thus $A = 3p$ and $B = -6p^2$. Do not divide out p just yet! There are two cases to consider:

- *If $p \neq 0$, A is nonzero and there is a unique solution: $S = \{\frac{-6p^2}{3p}\} = \{-2p\}$.*
- *If $p = 0$, both A and B are zero and any real number is a solution: $S = \mathbb{R}$. (We can check this by seeing that inserting $p = 0$ into the original equation yields the trivial $x = x$.)*

Example 2.8.

$$(c+3)x = c+3$$

Here $D = \mathbb{R}$ and $A = B = c+3$. Again two cases:

- *If $c+3 = 0$, i. e. $c = -3$, the equation simplifies to $0x = 0$ and thus $S = \mathbb{R}$.*
- *If $c \neq -3$, we can solve for x and get $x = \frac{c+3}{c+3} = 1$, so $S = \{1\}$.*

Example 2.9.

$$\begin{aligned} b(x-1) &= -(x+1) && D=\mathbb{R}, \textit{expand} \\ bx - b &= -x - 1 && +x+b \\ bx + x &= b - 1 && \textit{factor LHS} \\ (b+1)x &= b-1 && A = b+1, \quad B = b-1 \end{aligned}$$

- *If $b+1 = 0$, i. e. $b = -1$, the equation reads $0 = -2$ and so $S = \{\}$.*
- *If $b \neq -1$, we can solve for x: $S = \{\frac{b-1}{b+1}\}$.*

Example 2.10. *Solve the equation $qx - x = q^2 - 1$ for x over the natural numbers, where $q \in \mathbb{N}$.*

$$\begin{aligned} qx - x &= q^2 - 1 && D = \mathbb{N}, \ \textit{factor LHS} \\ (q-1)x &= q^2 - 1 && A = q-1, \quad B = q^2 - 1 \end{aligned}$$

- *If $q \neq 1$, A is nonzero and we can solve for x: $x = \frac{q^2-1}{q-1} = q+1$, so $S = \{q+1\}$.*
- *If $q = 1$, both A and B are zero and thus $S = \mathbb{N}$. (The equation is the trivial $0 = 0$.)*

Exercise 2.1. *Which of these equations are linear in x? (Solving them is not required.)*

(a) $4x - 1 = x$
(b) $3x \times 2x = 9x$
(c) $\dfrac{1-x}{2} = -\dfrac{7x}{3}$
(d) $\dfrac{x}{3} = \dfrac{5}{x}$
(e) $a^2 x = 5 + x$
(f) $\dfrac{x^2-1}{x-1} = \sqrt{7}$

Exercise 2.2. *Solve these equations.*

(a) $7x - (4x - 5) = 29$
(b) $2x - 18 = 5x + 1$
(c) $23x - 16 + 10(1 + x) = 4(5 + 6x) + 9x - 14$
(d) $x^2 + 17 = 49 - x(11 - x)$
(e) $1 - (2 - (3 - (4 - x))) = x$
(f) $\frac{4x + 1}{6} = -\frac{7x - 8}{8}$
(g) $\frac{18x}{63} - 0.6 = \frac{22x - 3}{77}$

Exercise 2.3. *Find the set of solutions depending on the value of the parameter.*

(a) $x(a - 3) = 2ax + 3$
(b) $p(x - 1) = x(p + 1)$
(c) $2x - xr = r^2 - 4,\ D = \mathbb{N}$